

INEQUALITIES OF LEVIN-STEČKIN, CLAUSING AND CHEBYSHEV REVISITED

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ABSTRACT. We prove the Levin-Stečkin inequality using Chebyshev's inequality and symmetrization. Symmetry and slightly modified Chebyshev's inequality are also the key to an elementary proof of Clausing's inequality .

1. INTRODUCTION

It seems that the Levin-Stečkin inequality appeared first in an appendix to the Russian edition of the famous Hardy, Littlewood and Pólya's Bible on inequalities [3]. The translator (Levin) enumerates the appendices written by Stečkin, by Levin and by both of them. The inequality we consider here comes from Appendix I written by Stečkin. But the English version of the appendix [4] did not probably make this distinction clear enough, so all inequalities cited in the literature are called Levin-Stečkin.

Theorem 1.1 (Levin-Stečkin's inequality). *Let the function $p: (0, 1) \rightarrow \mathbb{R}$ satisfies the conditions*

- (1) *p is non-decreasing in $(0, \frac{1}{2})$,*
- (2) *p is symmetric, i.e. $p(x) = p(1 - x)$,*

then for every convex function φ the inequality

$$(1.1) \quad \int_0^1 p(x)\varphi(x)dx \leq \int_0^1 p(x)dx \int_0^1 \varphi(x)dx.$$

The original proof is elementary, but quite complicated. Recently Mercer ([5]) published a proof that uses the notion of extremal points of the set of concave positive functions satisfying $\int_0^1 f(x)dx \leq 1$. His method, not very elementary, has an advantage: leads to a simple proof of the Clausing inequality.

Theorem 1.2 (Clausing's inequality [2]). *Let p be nonnegative functions on $(0, 1)$ satisfying the following conditions:*

- *p are symmetric (i.e. $p(x) = p(1 - x)$),*
- *p is non-decreasing on $[0, 1/2]$,*

Then for every concave, positive function φ the inequality

$$(1.2) \quad \int_0^1 p(x)\varphi(x)dx \leq \int_0^1 \varphi(x)dx \int_0^1 4 \min\{x, 1 - x\}p(x)dx$$

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holds.

Both inequalities make the reader think of the inequality of Chebyshev, linking the integral of a product of functions with the product of integrals.

Theorem 1.3 (Chebyshev's inequality). *If the functions $f, g : [a, b] \rightarrow \mathbb{R}$ are monotone in the same direction, then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \geq \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx$$

The inequality is reversed if the monotonicities are opposite.

Our aim is to give elementary proofs of Levin-Stečkin's and Clausen's inequalities. The proofs we offer here are sponsored by the word *symmetrization*.

2. THE LEVIN-STEČKIN INEQUALITY

We prove this inequality in two steps: firstly we show that Theorem 1.1 is valid for symmetric functions:

Lemma 2.1. *Under the assumptions of Theorem 1.1 if φ is symmetric and convex, then the inequality (1.1) holds.*

Proof. Suppose φ is convex. Its symmetry implies that it is non-increasing in the interval $(0, \frac{1}{2})$, and using Chebyshev's inequality we get

$$\begin{aligned} \int_0^1 p(x)dx \int_0^1 \varphi(x)dx &= \left(\int_0^{1/2} p(x)dx + \int_{1/2}^1 p(x)dx \right) \left(\int_0^{1/2} \varphi(x)dx + \int_{1/2}^1 \varphi(x)dx \right) \\ &= 4 \int_0^{1/2} p(x)dx \int_0^{1/2} \varphi(x)dx \geq 2 \int_0^{1/2} p(x)\varphi(x)dx = \int_0^1 p(x)\varphi(x)dx. \quad \square \end{aligned}$$

Now consider arbitrary φ .

Proof of the Levin-Stečkin inequality. Once more we shall explore the symmetry. Note that for convex φ the function $\frac{\varphi(x) + \varphi(1-x)}{2}$ is convex and symmetric, so we can use Lemma 2.1

$$\begin{aligned} \int_0^1 p(x)\varphi(x)dx &= \int_0^1 p(x) \frac{\varphi(x) + \varphi(1-x)}{2} dx \\ &\leq \int_0^1 p(x)dx \int_0^1 \frac{\varphi(x) + \varphi(1-x)}{2} dx \\ &= \int_0^1 p(x)dx \int_0^1 \varphi(x)dx. \end{aligned}$$

□

3. CHEBYSHEV'S INEQUALITY

To prove the Clausing inequality we need a bit stronger version of Chebyshev's inequality, where the monotonicity of one function get replaced by a weaker condition. Note that this result is somewhat similar to the result of Brunn [1].

Definition 3.1. We shall say that an integrable function $f : [a, b] \rightarrow \mathbb{R}$ belongs to the class M^+ if there is a $c \in [a, b]$ such that

- (1) if $f(x) < \frac{1}{b-a} \int_a^b f(x)dx$, then $x < c$ and
- (2) if $f(x) > \frac{1}{b-a} \int_a^b f(x)dx$, then $x > c$.

We say that f belongs to M^- if the inequalities in (1) and (2) are reversed.

Obviously every non-decreasing function belongs to the class M^+ and a non-increasing one is a member of M^- .

Theorem 3.1. If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable, g is non-decreasing and $f \in M^+$ or g is non-increasing and $f \in M^-$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \geq \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx.$$

Exchanging M^+ and M^- toggles the inequality.

Proof. Let $f \in M^+$ and g be non-decreasing (the proof in other cases is similar). Denote $f^* = \frac{1}{b-a} \int_a^b f(x)dx$. Then

$$\begin{aligned} \int_a^b [f(x) - f^*] g(x)dx &= \int_a^c [f(x) - f^*] g(x)dx + \int_c^b [f(x) - f^*] g(x)dx \\ &\geq g(c) \int_a^c [f(x) - f^*] dx + g(c) \int_c^b [f(x) - f^*] dx = 0. \quad \square \end{aligned}$$

4. CLAUSING'S INEQUALITY

In this section we present an elementary proof of a generalization of the Clausing inequality.

Theorem 4.1. Let p, q be nonnegative functions on $(0, 1)$ satisfying the following conditions:

- p and q are symmetric (i.e. $p(x) = p(1-x)$),
- p is increasing on $[0, 1/2]$,
- q is convex on $[0, 1/2]$,
- $q(0) = 0$ and $\int_0^1 q(x)dx = 1$.

Then for every concave function φ with $\varphi(0) + \varphi(1) \geq 0$ the inequality

$$(4.1) \quad \int_0^1 p(x)\varphi(x)dx \leq \int_0^1 \varphi(x)dx \int_0^1 p(x)q(x)dx$$

holds.

Proof. Assume first that φ is symmetric and denote $\int_0^1 \varphi(x)dx = K$. The inequality (4.1) can be rewritten as

$$(4.2) \quad 0 \leq \int_0^{1/2} [Kq(x) - \varphi(x)]p(x)dx.$$

The Hermite-Hadamard inequality yields $K \geq 0$ and the symmetry of φ implies $\varphi(0) \geq 0$, thus the function $Kq - \varphi$ is convex, $Kq(0) - \varphi(0) \leq 0$ and $\int_0^{1/2} [Kq(x) - \varphi(x)]dx = 0$, therefore it belongs to the class M^+ , and by Theorem 3.1 $\int_0^{1/2} [Kq(x) - \varphi(x)]p(x)dx \geq 0$ which proves (4.2).

Now let φ be arbitrary. We have

$$\begin{aligned} \int_0^1 p(x)\varphi(x)dx &= \int_0^1 p(x)\frac{\varphi(x) + \varphi(1-x)}{2}dx \\ &\leq \int_0^1 \frac{\varphi(x) + \varphi(1-x)}{2}dx \int_0^1 p(x)q(x)dx \quad (\text{by (4.1)}) \\ &= \int_0^1 \varphi(x)dx \int_0^1 p(x)q(x)dx \end{aligned}$$

which completes the proof. \square

The function $q_0(x) = 4 \min\{x, 1-x\}$ is a borderline between admissible q 's and sample functions φ . Setting $\varphi = q_0$ in (4.1) we obtain

$$\int_0^1 p(x)q_0(x)dx \leq \int_0^1 p(x)q(x)dx$$

which means that q_0 provides the best bound in (4.1).

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